Cluster realization of spherical DAHA

double affine Hecke algebra

(joint w/ P. Di Francesco, R. Kedem, G. Schrader)

Plan: $s\mathcal{H}_{q,t}$ — GL_n sDAHA

$\circlearrowleft$

$SL(2,\mathbb{Z})$

Thm: (Di Francesco – Kedem – Schrader – S. ' in progress)

1) $\exists\ \mathcal{X}_Q^q$ – quantum cluster variety & an injective homomorphism

$i : s\mathcal{H}_{q,t} \longrightarrow \mathcal{X}_Q^q$

isom?

| Q – cluster quiver

2) $SL_2(\mathbb{Z})$ – action is realized via cluster mutations

Idea:

"Polynomial" representation $\rho_1 : s\mathbb{H}_{q,t} \curvearrowright \mathcal{H}_1$

"Positive" representation $\rho_2 : \mathcal{X}^q_Q \curvearrowright \mathcal{H}_2$

$\mathcal{H}_1, \mathcal{H}_2$ — Hilbert spaces

ρ_1, ρ_2 — faithful, $SL(2,\mathbb{Z})$-equivariant

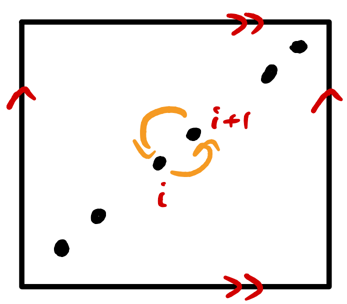
$$
\begin{tikzcd}
\rho_1 : & s\mathbb{H}_{q,t} \arrow[d, dashed, <->] & \curvearrowright & \mathcal{H}_1 \\
\rho_2 : & \mathcal{X}^q_Q & \curvearrowright & \mathcal{H}_2 \arrow[u, "\wr", "W"']
\end{tikzcd}
$$

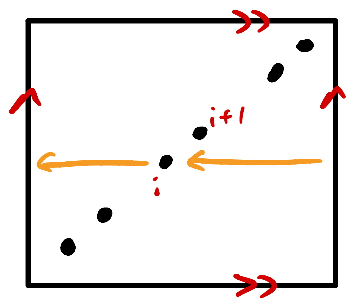
← spectral transform of q. Toda system

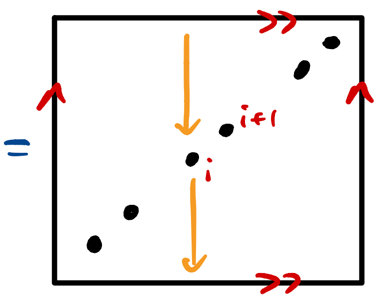
I. DAHA $(G = GL_n)$

$B_n^{ell} := \pi_1(\mathrm{Conf}_n(T^2))$ — elliptic Braid gp.

Generators: $T_1, \dots, T_{n-1}$; $X_1, \dots, X_n$; $Y_1, \dots, Y_n$.

$T_i =$ 

$X_i =$ 

$Y_i =$ 

Rel-s : . . .

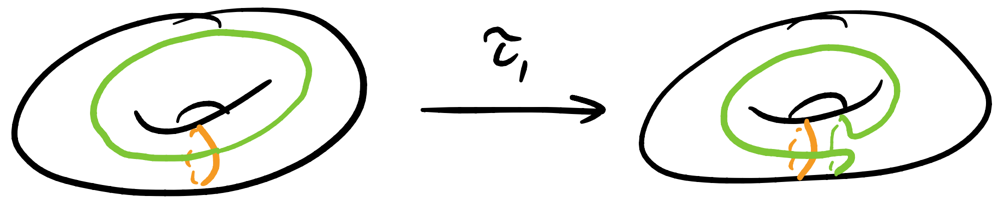
$$\mathcal{H}_{q,t} := \mathbb{C}(q,t)[B_n^{ell}] \Big/ \langle (T_i - q^{-1}t)(T_i + q\,t^{-1}) = 0 \rangle_{i=1}^{n-1}$$

$\uparrow$ DAHA

$SL(2,\mathbb{Z}) \circlearrowright \mathcal{H}_{q,t}$

$\Downarrow \quad \Downarrow$

$\tau_1, \ \tau_2 \leftarrow$ Dehn twists



$\tau_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $T_1 \longmapsto T_1$, $X_1 \longmapsto X_1$, $Y_1 \longmapsto Y_1 X_1$

$\tau_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$: $T_1 \longmapsto T_1$, $X_1 \longmapsto X_1 Y_1$, $Y_1 \longmapsto Y_1$

$\exists \, e \in \mathcal{H}_{q,t}$: $e^2 = e \leftarrow$ idempotent

$\tau_1 e = \tau_2 e = e$

$s\mathcal{H}_{q,t} := e\,\mathcal{H}_{q,t}\,e \leftarrow$ spherical DAHA

$\circlearrowright$ $SL(2,\mathbb{Z})$

$\Downarrow$ e $\nwarrow$ unit

$P_{0,k} := e(Y_1^k + \dots + Y_n^k)e$

$g \in SL(2,\mathbb{Z}),\ g\binom{0}{k} = \binom{a}{b} \implies P_{a,b} := g \circ P_{0,k}$

$\implies P_{k,0} = e(X_1^k + \dots + X_n^k)e$

Set $E_{k,0} := e\, e_k(x_1, \dots, x_n) e$

$E_{0,k} := e\, e_k(Y_1, \dots, Y_n) e$

$e_k(\bar{x}) = \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \dots x_{i_k}$

k-th elem sym. f-n

<u>Thm:</u> (Schiffmann - Vasserot)

$P_{a,b}$ generate $s\mathcal{H}_{q,t}$

<u>Polynomial rep-n.</u>

$\mathcal{H}_{q,t} \circlearrowright \mathbb{C}(q,t)[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}$

$E_{k,0} \longmapsto e_k(\bar{x})$

$E_{0,k} \longmapsto \sum_{\substack{I \subset \{1,\dots,n\} \\ |I| = k}} \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} \Gamma_i$

Macdonald operator

$\Gamma_i f(\bar{x}) = f(x_1, \dots, q x_i, \dots x_n)$

X_i acts by mult.

II. Quantum Toda chain

$G\ (=GL_n)$, $B_\pm$ — Borels, H — max torus

$G^{u,v} = B_+ u B_+ \cap B_- v B_-$ $\quad u,v \in W$ — Weyl gp

Classical limit:

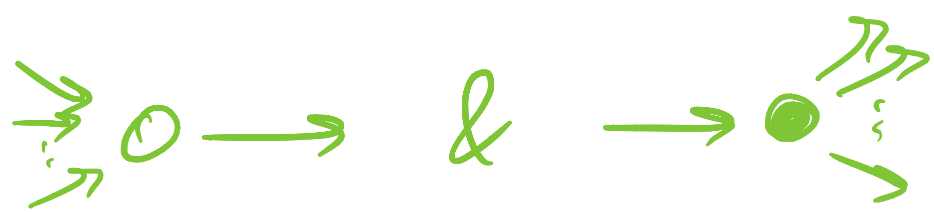
Coxeter word

phase space $= G^{c,c}/\mathrm{Ad}H$ $\quad$ (e.g. $c = s_1 s_2 \dots s_{n-1}$)

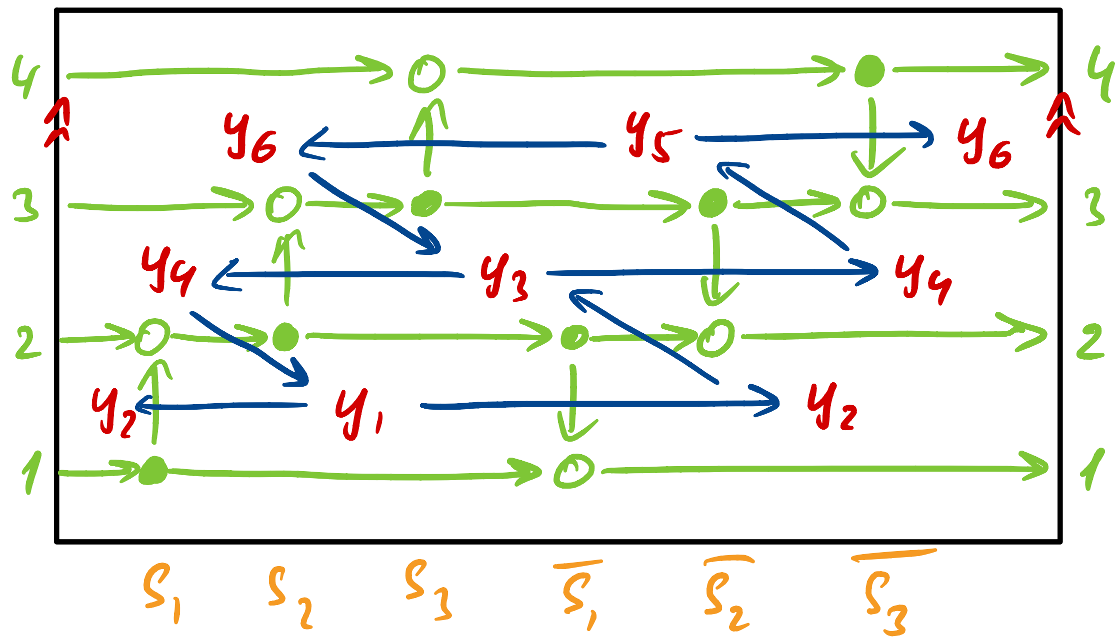
Hamiltonians: $H_i = \mathrm{tr}_{\Lambda^i \mathbb{C}^n}(g)$

Cluster structure:

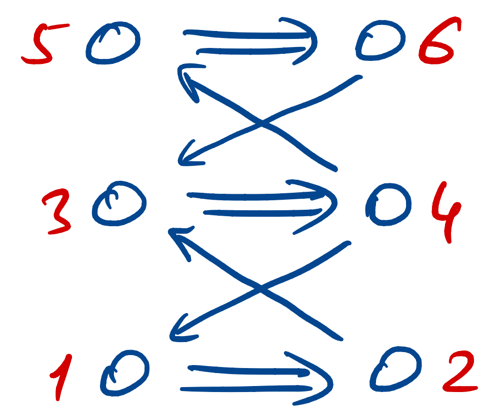
Directed network is a planar graph on a surface with vertices of two types:



Ex:



directed network on a cylinder



dual graph Q

$$M_{ij} := \sum_{p: i \to j} wt(p) \qquad wt(p) = \text{product of wts of faces below path } p \Big/ (\det M)^{1/n}$$

$$M_{11} = 1 + y_1$$

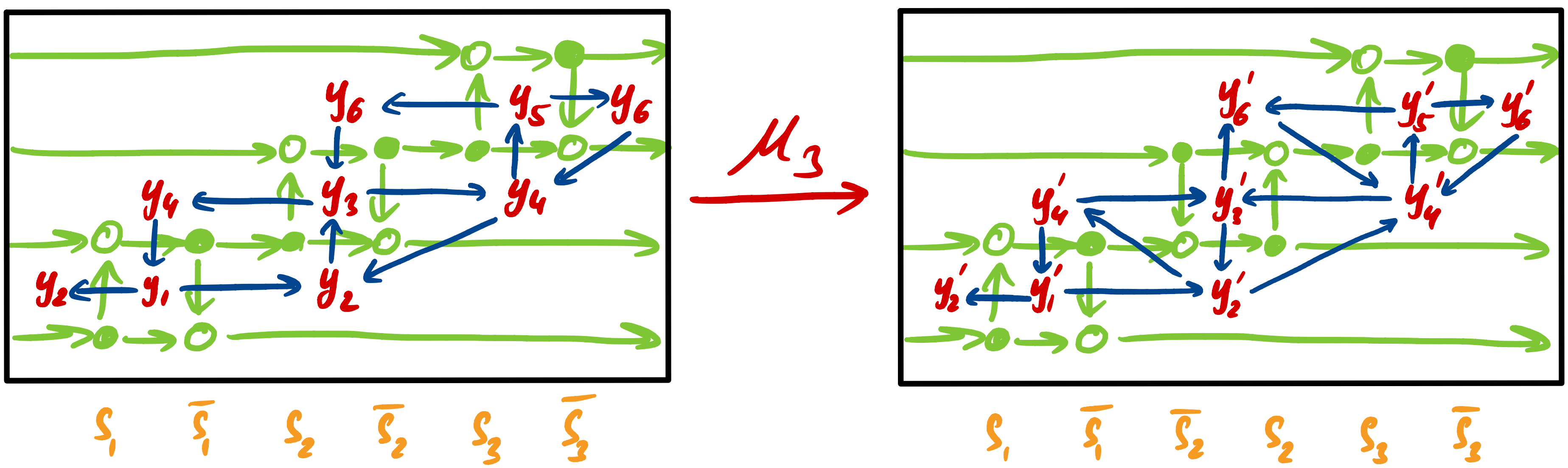
Parameterize $G^{c,c}/\mathrm{Ad}H \ni [g] = M$

$y_1, \dots, y_{2n-2}$ — cluster coord-s on $G^{c,c}/\mathrm{Ad}H$

$$\{y_j, y_k\} = \#\{j \to k\}\, y_j y_k$$

Cluster mutations:

$$s_1\, s_2\, s_3\, \bar{s}_1\, \bar{s}_2\, \bar{s}_3 \;=\; s_1 \bar{s}_1 \boxed{s_2 \bar{s}_2}\, s_3 \bar{s}_3 \xrightarrow{\mu_3} s_1 \bar{s}_1 \boxed{\bar{s}_2\, s_2}\, s_3\, \bar{s}_3$$



$(y_1', \ldots, y_6')$ are defined from $\mathcal{M} = \mathcal{M}'$:

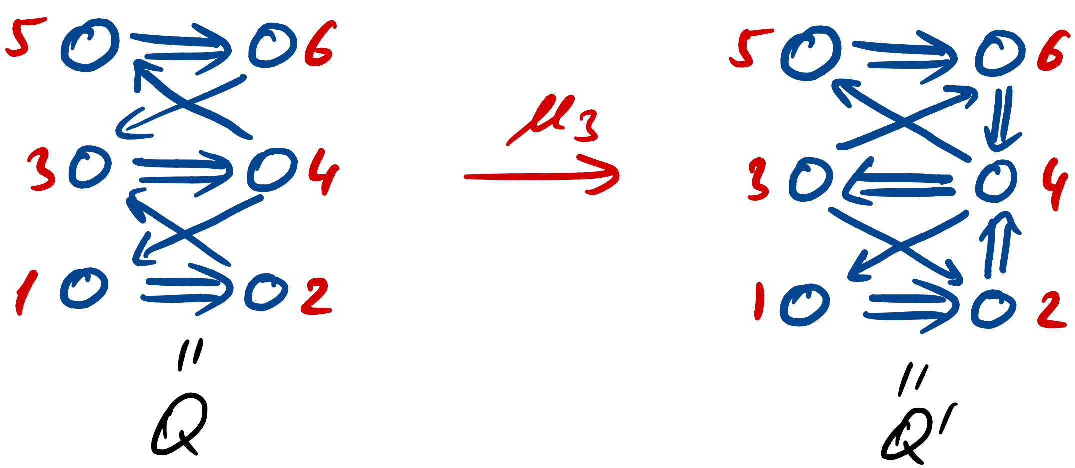
coord-s on a new cluster chart

$$y_1' = y_1 \qquad y_2' = y_2(1+y_3)$$

$$y_3' = y_3^{-1} \qquad y_4' = y_3 y_4 (1+y_3)^{-2}$$

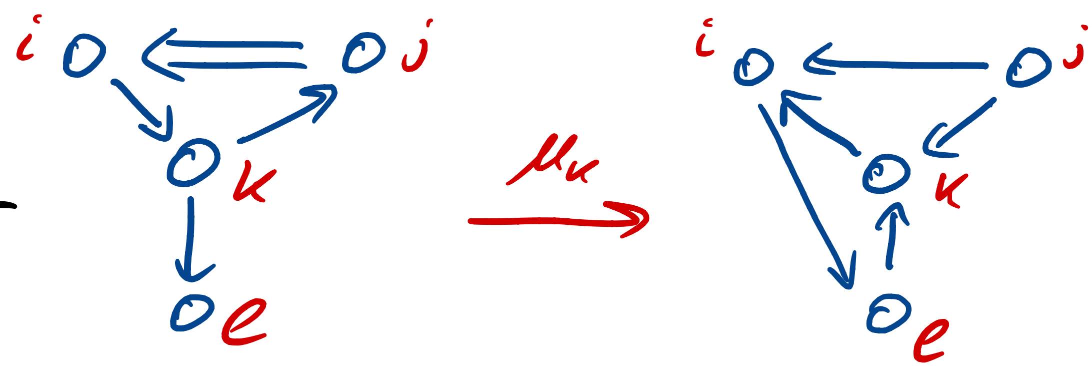
$$y_5' = y_5 \qquad y_6' = y_6(1+y_3)$$

Quivers:



In general: can mutate at any vertex

Ex:



$$y_k' = y_k^{-1}$$
$$y_i' = y_i(1+y_k)$$
$$y_j' = y_j y_k (1+y_k)^{-1}$$
$$y_e' = y_e y_k (1+y_k)^{-1}$$

Def: $\mathcal{O}(\mathcal{X})$ — global algebra of functions, i.e. expressions in $(y_1, \dots, y_n)$ Laurent in every chart

$$\mathcal{X} := \operatorname{Spec} \mathcal{O}(\mathcal{X})$$

Toda Hamiltonians:

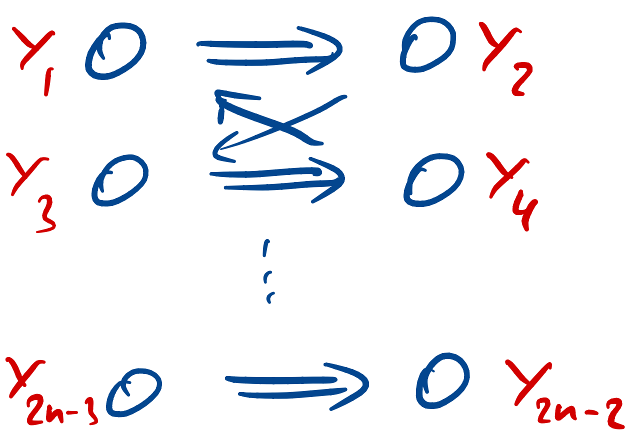
$$H_k = \sum_{\substack{I \subset \{1,\dots,n\} \\ |I| = k}} \sum_{\substack{\text{non-intersecting} \\ \text{paths } p_1,\dots,p_k : I \to I}} wt(p_1) \cdots wt(p_k)$$

e.g. $H_1 = tr(M)$

Lemma: $H_k \in \mathcal{O}(\mathcal{X}_Q)$

Quantum Toda: $y_i \longrightarrow Y_i$

$$Y_j Y_k = q^{\#(k \to j)} Y_k Y_j$$



$$\mathbb{C}(q)\langle Y_1, \dots, Y_n \rangle \hookrightarrow \mathbb{C}(q)[\Lambda_1^{\pm 1}, \dots, \Lambda_n^{\pm 1}]$$

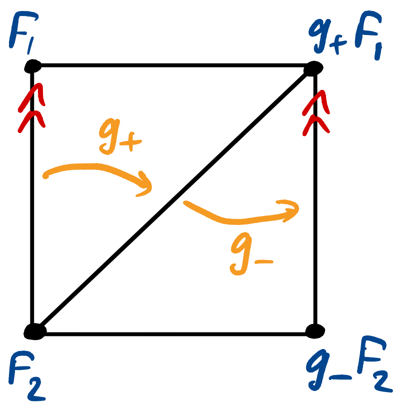
$$Y_{2i} \longmapsto T_{\alpha_i} := T_{i+1}/T_i$$

$$Y_{2i-1} \longmapsto \Lambda_{\alpha_i} := \Lambda_{i+1}/\Lambda_i$$

$$T_i f(\vec{\Lambda}) = f(\Lambda_1, \dots, q\Lambda_i, \dots, \Lambda_n)$$

Λ_i acts by mult.

Dehn twist



$\mathcal{M} :=$ Moduli space of G-local systems on a cylinder w/ flags F_1, F_2 at 2 marked pts.

$$g_+ F_2 = F_2$$

$$g_-(g_+ F_1) = g_+ F_1$$

$$g := g_- g_+ \implies \begin{cases} g F_1 = g_+ F_1 \\ g F_2 = g_- F_2 \end{cases}$$

Toda phase space $= \mathcal{M}$ w/ cond-s on relative positions of F_j & gF_j, $j=1,2$

$(g_+ \in G^{1,c},\ g_- \in G^{c,1})$

Dehn twist $=$ flip of diagonal

$\parallel$

D

$=$ refactorization $g = \tilde{g}_+ \tilde{g}_-$

$=$ cluster mutations at $y_1, y_3, \ldots y_{2n-3}$

$[D, H_j] = 0$

$=$ 1-step evolution of Q-system

III. Whittaker transform

Parameterize: $\boxed{q = e^{\pi i b^2}, \quad b \in \mathbb{R}}$

$$\boxed{\begin{aligned} \Lambda_j &= e^{2\pi b \lambda_j}, & T_j &= e^{-ib\,\partial/\partial\lambda_j} \\ X_j &= e^{2\pi b x_j}, & P_j &= e^{-ib\,\partial/\partial x_j} \end{aligned}}$$

q-Toda $\circlearrowright L^2(\bar{\lambda}, d\bar{\lambda}) =: \mathcal{H}_2$

sDAHA $\circlearrowright L^2_{sym}(\bar{x}, m(\bar{x})d\bar{x}) =: \mathcal{H}_1$

Whittaker f-n: $\Psi_{\bar{x}}(\bar{\lambda}) \longleftarrow$ joint eigenf-n for H_k & D

Whittaker transform:

$$W: \mathcal{H}_2 \longrightarrow \mathcal{H}_1$$

$$f(\bar{\lambda}) \longmapsto \hat{f}(\bar{x}) := \int f(\bar{\lambda}) \Psi_{\bar{x}}(\bar{\lambda})\, d\bar{\lambda}$$

Thm: (Schrader – S.)

W is a unitary equivalence

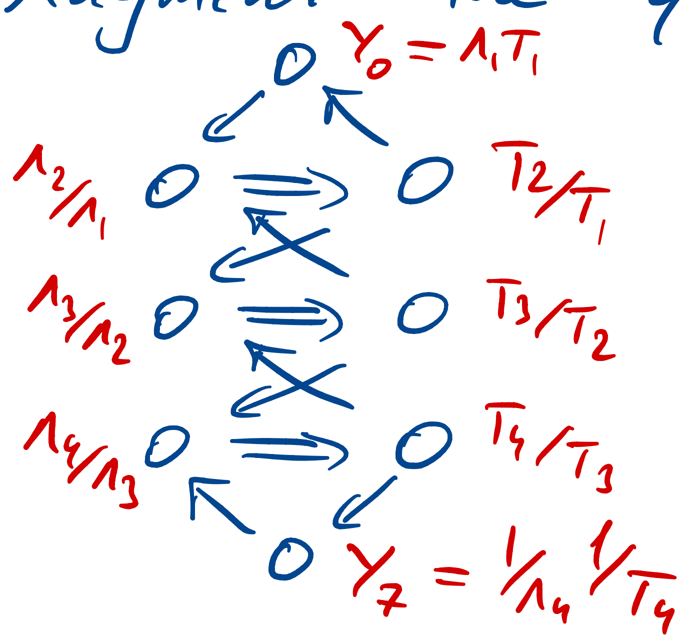
Moreover:

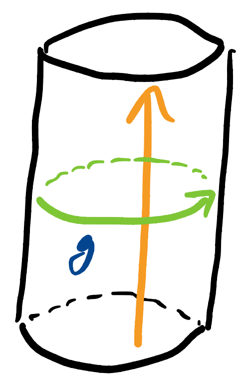
$$H_k \,\Psi_{\bar{x}}(\bar{\lambda}) = e_k(X_1, \dots, X_n)\, \Psi_{\bar{x}}(\bar{\lambda})$$

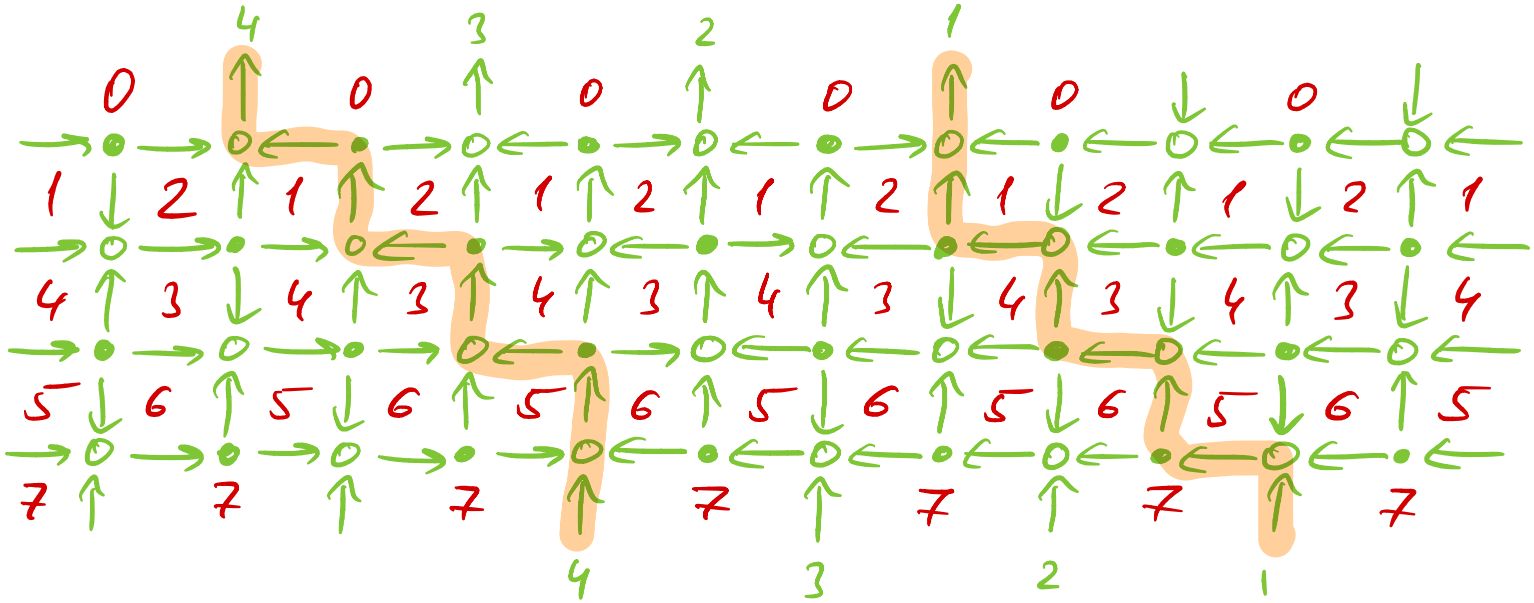
$$D\, \Psi_{\bar{x}}(\bar{\lambda}) = \tau_1 \Psi_{\bar{x}}(\bar{\lambda})$$

$\Rightarrow$ got half of DAHA

Now: Augment the quiver & consider the other monodromy







$\mathcal{M} = (\mathcal{M}_{ij}) \leftarrow$ monodromy matrix

$\lambda = (\lambda_1 \le \lambda_2 \le \ldots \le \lambda_k) \qquad \mu = (\mu_1 \le \mu_2 \le \ldots \le \mu_k)$

$\mathcal{M}_{\lambda\mu}$ - submatrix at the intersection of rows $(\lambda_1+1, \ldots, \lambda_k+k)$ & columns $(\mu_1+1, \ldots, \mu_k+k)$

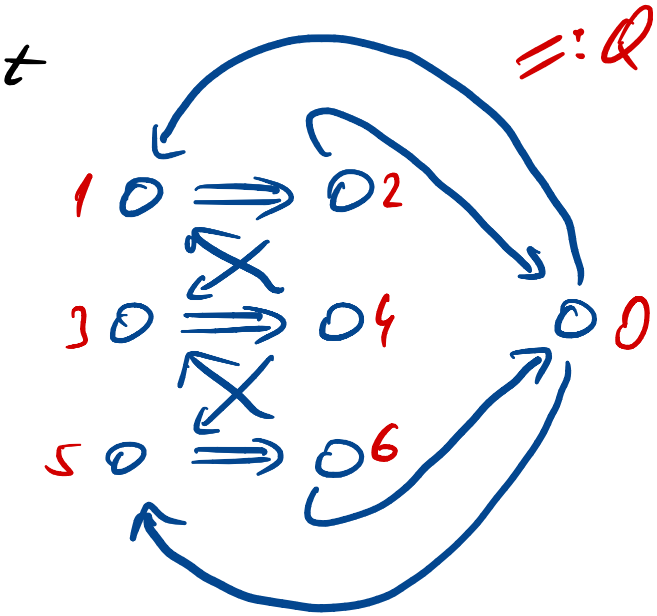
$$\Rightarrow \quad \det(\mathcal{M}_{\lambda\mu}) \xmapsto{\mathcal{W}} \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I| = k}} s_{\lambda'}(\bar{x} \setminus \bar{x}_I)\, s_\mu(\bar{x}_I) \prod_{\substack{i\in I \\ j \notin I}} \frac{1}{x_i - x_j} \prod_{i \in I} \Gamma_i$$

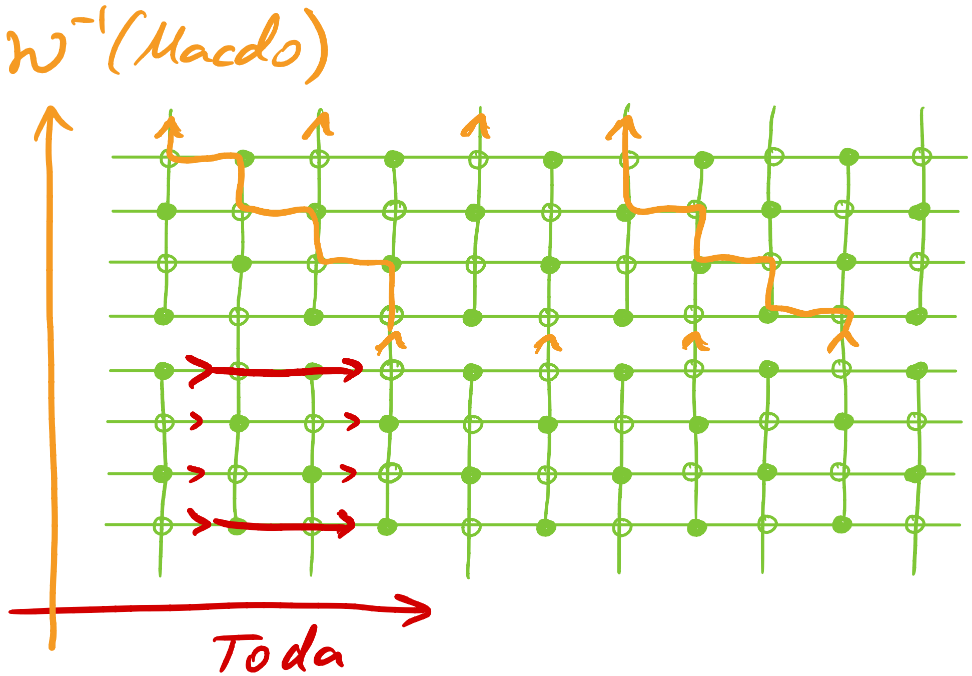
conjugate partition

$\Rightarrow$ allows to calculate $\mathcal{W}^{-1}(P_{a,b}) \in \mathcal{X}_Q^q \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$

IV. Cluster realization of $s\mathcal{H}_{q,t}$

Glue top & bottom vertices & put quiver on a torus





$$Y_0 = {}^{\Lambda_1}\!/_{\Lambda_n}\ {}^{T_1}\!/_{T_n}\ t$$

$$H_k = \sum_{\substack{\text{non-intersect} \\ \text{paths } p_1 \dots p_k}} wt(p_1) \dots wt(p_k)$$

$$\Delta_k := \sum_{\substack{\text{non-intersect} \\ \text{paths } p_1 \dots p_k}} wt(p_1) \dots wt(p_k)$$

$$\Longrightarrow \begin{cases} H_k \Psi = E_{k,0} \Psi \\ \Delta_k \Psi = E_{0,k} \Psi \end{cases}$$

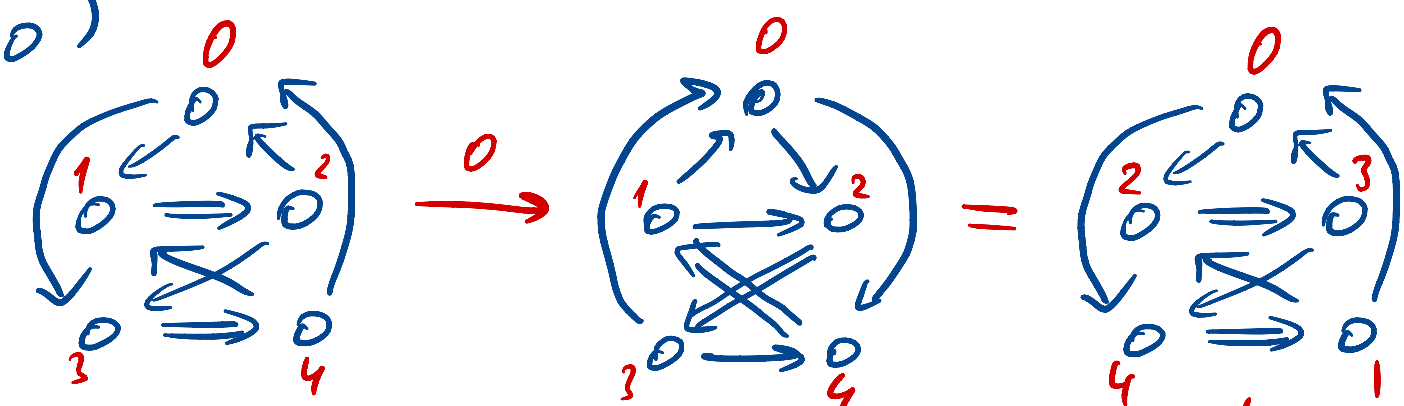
$SL(2,\mathbb{Z})$: $\tau_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ = mutate at sources of double arrows.

$\tau_2 = ?$

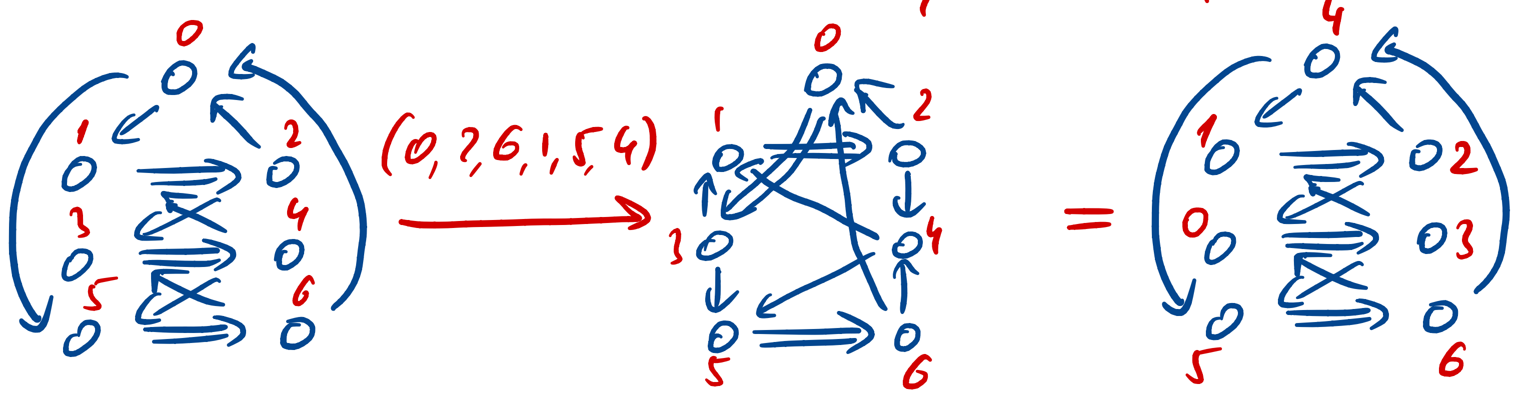
Work in progress: Each el-t of $SL(2,\mathbb{Z})$ is realized via a cluster transformation.

Ex: $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$n=3$



$n=4$

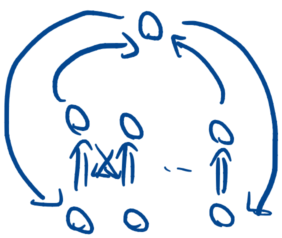


V. Conclusion

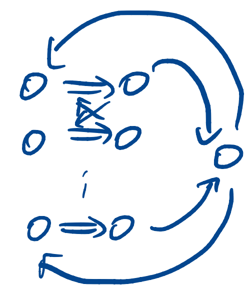
Cluster structure on quantized moduli space of degenerate local systems on punctured torus

positive rep-n

positive rep-n



$$\left\{\begin{array}{l} E_{k,0} = H_k \ \uparrow \\ E_{0,k} = \Delta_k \ \rightarrow \end{array}\right\} \xleftrightarrow{\text{cluster mut-s}} \left\{\begin{array}{l} E_{k,0} = \Delta_k \ \uparrow \\ E_{0,k} = H_k \ \rightarrow \end{array}\right\}$$



$$\Big\downarrow W\uparrow \qquad\qquad\qquad\qquad \Big\downarrow W\rightarrow$$

$$\left\{\begin{array}{l} E_{k,0} = e_k(X_1,\dots,X_n) \\ E_{0,k} = \mathrm{Macdo}(\bar{X}) \end{array}\right\} \qquad \left\{\begin{array}{l} E_{k,0} = \mathrm{Macdo}(\bar{Y}) \\ E_{0,k} = e_k(Y_1,\dots,Y_n) \end{array}\right\}$$

polynomial rep-n

polynomial rep-n

$$S\mathcal{H}_{q,t}$$